MATH 2050C Lecture 3 (Jan 17)

Goal: $\mathbb{R}$ is a complete ordered field.
today

Def^{n}: (Absolute value) Let $a \in \mathbb{R}$.

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note: $|a| \geq 0 \quad \forall a \in \mathbb{R}$

Prop: (a) $|ab| = |a| \cdot |b|$

(b) $|a|^2 = a^2$

*(c) Let $c \geq 0$. Then $|a| \leq c \iff -c \leq a \leq c$

(d) $-|a| \leq a \leq |a|$

Proof: (a) We exhaust all possible cases from Trichotomy (O2).

Case 1: Either a or b is 0.

Then, $ab = 0 \Rightarrow |ab| = 0$.

Also, if $a = 0$, then $|a| = 0 \Rightarrow |a| \cdot |b| = 0$.

Same for $b = 0$. So, $|ab| = |a||b| = 0$.

Case 2: $a > 0$ and $b < 0$.

Then, by Prop. last time, $ab < 0 \Rightarrow |ab| = -ab$.

Also, $\left.\begin{array}{l} a > 0 \Rightarrow |a| = a \\ b < 0 \Rightarrow |b| = -b \end{array}\right\}$ $|a| \cdot |b| = a \cdot (-b) = -ab$. // Same

Ex: Check this!

Case 3: $a > 0$ and $b > 0$

Case 4: $a < 0$ and $b < 0$ $\Big\}$ Ex:

Same as Case 2 → **Case 5:** $a < 0$ and $b > 0$

(b) Take $b = a$ in (a),

$$a^2 = |a^2| = |ab| = |a||b| = |a|\cdot|a| = |a|^2.$$

$\because a^2 \geq 0 \quad \forall a \in \mathbb{R}$.

(c) Exhaust all cases of a by trichotomy (Ex:)

(d) Follows from (c) by taking $c = |a| \geq 0$. $\blacksquare$

Some Useful Inequalities

(1) AM-GM inequality: $\underbrace{\sqrt{ab}}_{\geq 0} \leq \frac{1}{2}(a+b) \quad \forall a, b \geq 0$

(2) Triangle inequality: $|a+b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$

(3) Bernoulli's inequality: $\underbrace{(1+x)^n}_{\geq 0} \geq 1 + n\cdot x \quad \forall x > -1, \forall n \in \mathbb{N}$

Proof: (1) Let $a, b \geq 0$, then $\sqrt{a}, \sqrt{b}$ exist (Assume this).

By previous lemma,

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = \underbrace{(\sqrt{a})^2}_{\text{// def}^{\underline{n}}} - 2\sqrt{a}\sqrt{b} + \underbrace{(\sqrt{b})^2}_{\text{// def}^{\underline{n}}}$$

$$= a - 2\sqrt{a}\sqrt{b} + b$$

Rearranging gives the desired inequality.

(2) By (d) above, we have

$$\left.\begin{array}{l} -|a| \leq a \leq |a| \\ -|b| \leq b \leq |b| \end{array}\right\} \xRightarrow{\text{add}} -(|a|+|b|) \leq a+b \leq |a|+|b|$$

$$\xRightarrow{(c)} \quad |a+b| \leq |a| + |b|.$$

(3) Induction on n.

$\underline{n=1}$: Trivial since $(1+x)^n = 1+x = 1+n\cdot x$. when $n=1$.

Assume $\underline{n=k \text{ is true}}$, then for $n = k+1$,

$$(1+x)^{k+1} = \underbrace{(1+x)}_{\substack{\vee \\ 0}} (1+x)^k$$

$\because x > -1 \qquad \geq \quad n = k$

$$\geq (1+x)(1+k\cdot x) \qquad \left(\because n=k \text{ is true and } x>-1\right)$$
$$= 1+(k+1)x + k\cdot x^2$$
$$\geq 1+(k+1)x \qquad (\because k>0,\ x^2\geq 0)$$

By M.I., we are done. $\square$

<u>Remark</u>: Let $a, b \geq 0$. Then

$$a \leq b \iff a^2 \leq b^2 \iff \sqrt{a} \leq \sqrt{b}.$$

<u>Prop</u>: (Reversed Triangle Ineq.)

$$\big||a|-|b|\big| \leq |a-b| \qquad \forall a,b\in\mathbb{R}.$$

<u>Pf</u>: Tutorial.

<u>Def/ Thm</u> (Completeness Property of $\mathbb{R}$)

Every $\phi \neq S \subseteq \mathbb{R}$ that has an "upper bound" ? must have a "supremum" ? in $\mathbb{R}$.

We first make sense of the ?'s.

<u>Def^{n}</u>: Let $\phi \neq S \subseteq \mathbb{R}$.

(a) S is bounded above if $\exists\, u\in\mathbb{R}$ s.t. $s \leq u \quad \forall s\in S$

Any such $u\in\mathbb{R}$ is called an upper bound of S.

(b) S is bounded below if $\exists\, w\in\mathbb{R}$ s.t. $s\geq w \quad \forall s\in S$

Any such $w\in\mathbb{R}$ is called a lower bound of S.

(c) S is bounded if it is both bdd above AND below.

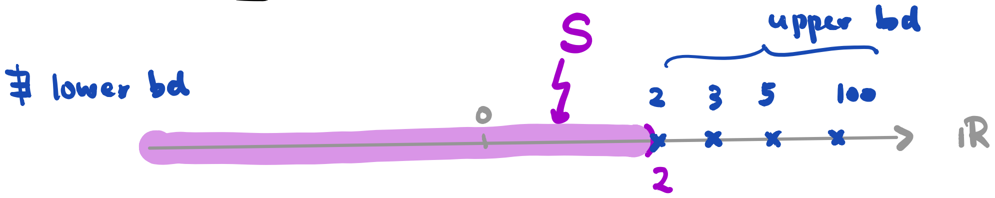
Otherwise, S is unbounded.

Example: $S := \{ x \in \mathbb{R} \mid x < 2 \}$

Note: There are many upper bds, e.g. $2, 3, 5, 100, \sqrt{100}$ etc...

$\Rightarrow$ S is bdd above.

But S is NOT bdd below. (Ex: prove it)



Def^{n}: Let $\phi \neq S \subseteq \mathbb{R}$.

(a) Suppose S is bdd above.

Then, $u \in \mathbb{R}$ is called a supremum (or least upper bound) of S if the following holds:

(i) u is an upper bd of S

(ii) $u \leq v$ for any upper bd v of S

[Notation: $u = \sup S$ or $\text{l.u.b.} S$]

(b) Similarly, we can define infimum (or greatest lower bound)

[Notation: $\inf S$ or $\text{g.l.b.} S$] Ex: Write this down.

Lemma: $\sup S$, if exists, is unique.

Proof: Suppose there are two $u, w \in \mathbb{R}$ which are supremum of S

Therefore, u, w satisfy (i), (ii) in the def^{n} above.

By (i) for w and (ii) for u, we have

$$u \leq w$$ $\because w$ is an upper bd

Similarly, by (i) for u and (ii) for w, we have

$$w \leq u$$ ← ∵ u is an upper bd.

Thus, $u = w$. $\square$

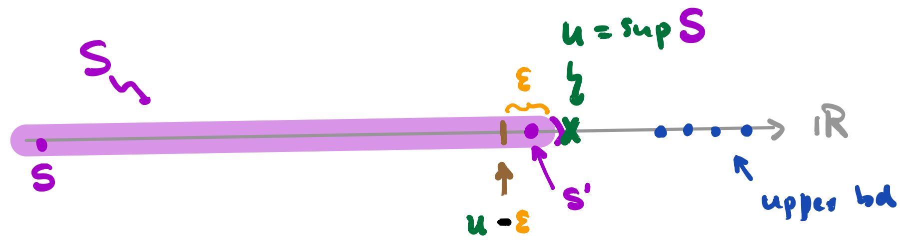
Prop: Let $\phi \neq S \subseteq \mathbb{R}$. Then $u = \sup S$ iff

(i) $s \leq u \quad \forall\, s \in S$

(ii) $\forall\, \varepsilon > 0,\ \exists\, s' \in S$ s.t. $u - \varepsilon < s'$

← (Useful way to prove $u = \sup S$)

Picture:



Proof: "⟹" Suppose $u = \sup S$.

By (i), u is an upper bd of S

$\Rightarrow u \geq s \quad \forall\, s \in S$ which is (i).

By (ii), $\underline{u \leq v \text{ for any upper bd. } v \text{ of } S}$. (*)

Fix $\varepsilon > 0$, but arbitrary.

Since $u - \varepsilon < u$, (*) $\Rightarrow$ $u - \varepsilon$ cannot be an upper bd.

So, $\exists\, s' \in S$ s.t. $u - \varepsilon < s'$

"⟸" Exercise. $\square$

Examples:

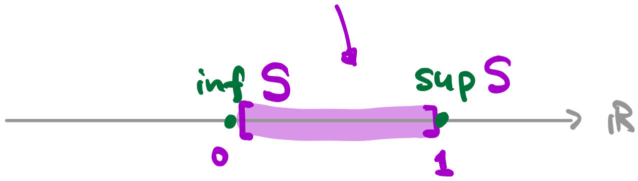
1) $S = \{s_1, \dots, s_n\}$ "finite set" (Assume: $s_1 < s_2 < \dots < s_n$)

$\sup S = s_n$

$\inf S = s_1$

(Ex: Prove.)

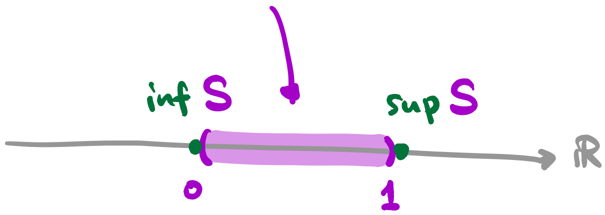
2) $S = [0, 1]$



$\sup S = 1 \in S$

$\inf S = 0 \in S$

3) $S = (0, 1)$



$\sup S = 1 \notin S$

$\inf S = 0 \notin S$